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Coulomb blockade and quantum tunnelling in an array of metallic grains

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Abstract

We study the effects of Coulomb interaction and inter-grain quantum tunnelling in an array of metallic grains using the phase-functional approach for temperatures T well below the charging energy E_c of individual grains yet large compared to the level spacing in the grains. When the inter-grain tunnelling conductance $g \gg 1$, the conductivity σ in d dimensions decreases logarithmically with temperature ($\sigma/\sigma_0 \sim 1 - \frac{1}{\pi g d} \ln(g E_c/T)$) (Panyukov and Zaikin 1991 *Phys. Rev. Lett.* **67** 3168, Goppert and Grabert 2000 *Eur. Phys. J.* B **16** 687, Efetov and Tschersich 2002 *Europhys. Lett.* **59** 114), while for $g \to 0$, the conductivity shows simple activated behaviour ($\sigma \sim \exp(-E_c/T)$). We show, for bare tunnelling conductance $g \gtrsim 1$, that the parameter $\gamma \equiv g(1 - 2/(g\pi)\ln(g E_c/T))$ determines the competition between charging and tunnelling effects. At low enough temperatures in the regime $1 \gtrsim \gamma \gg 1/\sqrt{\beta E_c}$, a charge is shared among a finite number $N = \sqrt{(E_c/T)/\ln(\pi/2\gamma z)}$ of grains, and we find a soft activation behaviour of the conductivity, $\sigma \sim z^{-1}\exp(-2\sqrt{(E_c/T)\ln(\pi/2\gamma z)})$, where z is the effective coordination number of a grain.

1. Introduction

Coulomb effects and electron tunnelling, as well as various effects of disorder, are major themes of mesoscopic physics. For a single metallic grain, all these effects have been explored extensively. For an array of normal metal grains, we find a peculiar interplay of electron charging and tunnelling effects manifested by a formation of a multi-grain charge excitation.

The novelty of granular metal systems in comparison with disordered metals or semiconductors with impurities arises from the presence of additional energy scales—the grain charging energy E_c [4] and intra-grain energy-level spacing δ. When the temperature is lowered below level spacing δ, electrons in the granular metal can propagate (diffuse or hop over many grains) coherently just like in a disordered metal. In the incoherent regime

$T > \max(g\delta, \delta)$, only charging effects and nearest-neighbour hopping (in the second order of the tunnelling matrix element) are relevant, and it is possible to formulate the problem in terms of longitudinal electromagnetic phase (or voltage) fluctuations on the grains [3, 5]. For large inter-grain conductance $g \gg 1$, the conductivity decreases logarithmically with temperature independent of dimensionality [1–3], reminiscent of many experiments [6–8]. The essential physics of this result was recognized some time ago as washed out Coulomb blockade for the quantum dot coupled to a conductive environment [9, 10]. For intermediate conductance $g \gtrsim 1$ (and sufficiently low temperature), we derive within the same approach a soft activation conduction ($\sigma \sim \mathrm{e}^{-\sqrt{T_0/T}}$) by a charge excitation (later referred to as a 'puddle') shared between many grains due to incoherent tunnelling. In the itinerant (or large-scale diffusion over many grains) regime at low temperatures $T < \max(g\delta, \delta)$, the granular metal for large conductance $g \gg 1$ is naturally described by Altshuler–Aronov theory [11, 12].

We employ the phase functional approach suggested over two decades ago by Ambegaokar, Eckern and Schön (AES) [5]. The original model described the tunnelling dynamics of granular superconductors, but nowadays this approach is increasingly used to study granular metals as well [3, 12, 13]. The AES action S_{AES} for granular metals consists of two contributions, $S_{\mathrm{AES}} = S_{\mathrm{c}} + S_{\mathrm{t}}$, where

$$S_{\mathrm{c}} = \frac{1}{2} \sum_{\mathbf{i},\mathbf{j}} \int_0^\beta \mathrm{d}\tau \, C_{\mathbf{ij}} \frac{\mathrm{d}\tilde{\phi}_i(\tau)}{\mathrm{d}\tau} \frac{\mathrm{d}\tilde{\phi}_j(\tau)}{\mathrm{d}\tau} \tag{1}$$

represents charging of the grains, and

$$S_{\mathrm{t}} = \pi g \sum_{|\mathbf{i}-\mathbf{j}|=a} \int_0^\beta \mathrm{d}\tau \, \mathrm{d}\tau' \, \alpha(\tau - \tau') \sin^2\left(\frac{\tilde{\phi}_{\mathbf{ij}}(\tau) - \tilde{\phi}_{\mathbf{ij}}(\tau')}{2} \right) \tag{2}$$

represents tunnelling between neighbouring grains, $\tilde{\phi}_{\mathbf{ij}} = \tilde{\phi}_{\mathbf{i}} - \tilde{\phi}_{\mathbf{j}}$. The kernel $\alpha(\tau)$ has the form $\alpha(\tau) = T^2(\mathrm{Re}(\sin(\pi T \tau + \mathrm{i}\epsilon))^{-1})^2$. The fields $\{\tilde{\phi}_{\mathbf{i}}\}$ are electromagnetic phase fluctuations on the grains related to the respective potential fluctuations $\{V_{\mathbf{i}}\}$ through $V_{\mathbf{i}}(\tau) = \partial_\tau \tilde{\phi}_{\mathbf{i}}(\tau)$. They satisfy bosonic boundary conditions, $\tilde{\phi}_{\mathbf{i}}(\tau) = \frac{2\pi k_{\mathbf{i}}}{\beta}\tau + \phi_{\mathbf{i}}(\tau)$, $\phi_{\mathbf{i}}(\tau) = \phi_{\mathbf{i}}(\tau + \beta)$, where the winding number $k_{\mathbf{i}}$ is an integer, and $-\infty < \phi_{\mathbf{i}}(\tau) < \infty$. The tunnelling conductance g is related to the inter-grain hopping amplitude $t_{\mathbf{i},\mathbf{i+a}}$ through $g = 2\pi |t_{\mathbf{i},\mathbf{i+a}}|^2/\delta^2$. Conductivity in the AES model is a second order (in hopping amplitude) incoherent tunnelling process between neighbouring grains. The elastic tunnelling lifetime τ of the electron on the grain is $\tau = \hbar/(g\delta)$. The condition, defining the granularity of the material and allowing averaging over fermionic intra-grain states, is that the tunnelling lifetime τ is much longer than the Thouless diffusion time l^2/D (where D is an intra-grain diffusion coefficient and l is the size of a grain). Another relevant condition is the implicit requirement of energy relaxation in the grains. The characteristic times associated with these incoherent dissipative processes should be shorter than tunnelling lifetime, consequently coherent combination of wavefunctions over grains cannot be written. Moreover, our diagrammatic analysis shows (see also [13]) that higher order processes ($|t_{\mathbf{i},\mathbf{i+a}}|^4$, etc) can be neglected when $T > g\delta$. Such a condition allows us to neglect 'dressing' of the tunnelling vertex $t_{\mathbf{i},\mathbf{i+a}}$ by further tunnelling lines. Therefore, the phase functional approach for granular materials can be justified if $g\delta \ll D/l^2$, and the temperature is sufficiently high, $T \gg \max(\delta, g\delta)$. We shall restrict our analysis to this regime.

2. Analysis of the model

The AES action shows important qualitative changes in the relevance of large phase fluctuations as the coupling g is varied. Consider the metallic phase $g \gg 1$ in equation (2). Expanding

$\sin^2(\tilde{\phi}_{\mathbf{ij}}(\tau) - \tilde{\phi}_{\mathbf{ij}}(\tau'))$ in a power series, we observe $\langle\tilde{\phi}^2_{\mathbf{ij}}\rangle \sim g^{-1}$, thus inter-grain phase fluctuations are Gaussian, and suppressed. The charge on an individual grain is not a well defined quantity, rather it is shared by the entire system. As g is progressively reduced, the phase fluctuations in equation (2) increase until finally one needs to take into account non-zero winding numbers $k_{\mathbf{i}} \neq 0$. In the extreme limit of $g \to 0$, the AES model describes a system of weakly coupled capacitors. The phase fluctuations are large, however the charge on an individual grain is well-defined. Conduction now involves exciting a charge which results in an activated temperature dependence of conductivity ($\sigma \sim g\exp(-\beta E_c)$). Such considerations lead us to examine whether for intermediate coupling between the grains, charge could be shared by a finite number of grains. This would be an intermediate situation between the extreme cases discussed above. In the remaining part of this paper, we choose a diagonal capacitance matrix in equation (1), $C_{\mathbf{ij}} \approx \frac{1}{2E_c}\delta_{\mathbf{ij}}$, to keep our analysis simple.

We describe now the physical picture for the soft activation phase which emerges from our analysis. Putting a single electron on an isolated grain costs E_c, while incoherent tunnelling enables the charge to be shared between two or more grains. We show below that the parameter $\gamma \equiv g(1 - 2/(\pi g)\ln(g\beta E_c))$ controls the suppression of winding numbers, and determines the degree of charge delocalization. When $\gamma \gtrsim 1$, the charge is effectively delocalized over the entire system (the charging energy is exponentially suppressed). For $\gamma \lesssim 1$, a unit of charge (electron) is shared among a finite number of grains. For simplicity we consider two grains, and compare statistical weights associated with the charge localized on any single grain $P_1 \sim \exp(-\beta E_c)$, and the charge shared between the two grains, $P_2 \sim \gamma\exp(-\beta E_c/2)$ ($\beta = 1/T$). Observe that the charging energy is halved upon hybridization[1]. Since a charge is shared classically (incoherently) between two grains, it is equivalent to equal average voltage on the grains, and thus two capacitors connected in parallel. The total capacitance is doubled, and the charging energy is halved. If $\gamma < \exp(-\beta E_c/2)$, the charge is unlikely to be shared between the two grains. If on the other hand, $\gamma > \exp(-\beta E_c/2)$, the electron is more likely to live on both the grains. Thus the two-grain hybridization 'puddle' optimizing the charging and tunnelling energies is formed. The optimum number N_* of hybridized grains sharing a single charge is determined by maximizing $P_N \sim \gamma^{N-1}\exp(-\beta E_c/N)$, which gives $N_* \sim \sqrt{\beta E_c/\ln(\gamma^{-1})}$, hence $\sigma \propto gP_{N_*} \sim \exp(-2\sqrt{\beta E_c\ln(\gamma^{-1})})$. This in essence is our main result.

To calculate the conductivity, we use Kubo's formula [3],

$$\sigma(\omega) = \frac{\mathrm{i}a^{2-d}}{\omega}\pi g\int_0^\beta \mathrm{d}\tau\,\alpha(\tau)(1 - \mathrm{e}^{\mathrm{i}\Omega_n\tau})\langle\cos(\tilde{\phi}_{\mathbf{i},\mathbf{i+a}}(\tau) - \tilde{\phi}_{\mathbf{i},\mathbf{i+a}}(0)\rangle|_{\Omega_n\to -\mathrm{i}\omega+\epsilon}, \tag{3}$$

where $\Omega_n = \frac{2\pi}{\beta}n$. Also of interest is the tunnelling density of states $\nu_{\mathbf{i}}(\varepsilon)$ into the grain $\mathbf{i}$:

$$\frac{\nu_{\mathbf{i}}(\varepsilon)}{\nu_0 T} = \mathrm{Im}\left[\int \mathrm{d}\tau\,\frac{\mathrm{e}^{\mathrm{i}\varepsilon_n\tau}}{\sin(\pi\tau T)}\tilde{\Pi}_{\mathbf{i}}(\tau)\bigg|_{\varepsilon_n\to -\mathrm{i}\varepsilon+\delta}\right], \tag{4}$$

where

$$\tilde{\Pi}_{\mathbf{i}} = \langle\exp(-\mathrm{i}(\tilde{\phi}_{\mathbf{i}}(\tau) - \tilde{\phi}_{\mathbf{i}}(0)))\rangle, \qquad \text{and} \qquad \varepsilon_n = \frac{2\pi}{\beta}\left(n + \frac{1}{2}\right).$$

At this stage we are in a position to understand qualitatively the logarithmic temperature dependence of the conductivity for $g \gg 1$ (derived in [1–3]). Since in this regime phase fluctuations are small, we set all $k_{\mathbf{i}} = 0$ and expand S_t to quartic order in $\phi_{\mathbf{ij}}$. Denoting the

[1] This incoherent hybridization of charge density between grains has to be distinguished from wavefunction hybridization (linear combination) in coherent, non-dissipative quantum mechanical systems.

Gaussian part of the resulting AES action as the 'free' action, and considering the quartic bit as 'interaction', one finds that the interaction renormalizes the tunnelling conductance as [3]

$$g_{\rm ren}(\tau-\tau') \approx g(1-\langle\phi_{\mathbf{ij}}(\tau)\phi_{\mathbf{ij}}(\tau')\rangle), \tag{5}$$

thus in d dimensions, $g_{\rm ren}(\beta) \approx g(1-\frac{1}{\pi g d}\ln(g\beta E_{\rm c}))$. One infers from equation (5) a similar temperature dependence for the conductivity σ as it is proportional to the effective tunnelling conductance $g_{\rm ren}$. This result suggests that when temperature T falls below $T_0 = E_{\rm c}{\rm e}^{-\pi d(g-1)}$, a transition or crossover into an insulating phase might be expected. Note that the parameter γ becomes smaller than one at temperature $T_1 = E_{\rm c}{\rm e}^{-\pi(g-1)/2}$, which is parametrically much larger than T_0. We show that T_1 marks the onset of soft activation behaviour.

Consider the AES model, equations (1) and (2). At low temperatures and for $g \gtrsim 1$, phase fluctuations on the grains could be large and non-Gaussian, so we expand the action about the finite winding number phase changes $(2\pi/\beta)k_{\mathbf{i}}\tau$ and residual fluctuations $\phi_{\mathbf{i}}$ [14]:

$$\begin{aligned} S[\{k_{\mathbf{i}}\};\{\phi_{\mathbf{i}}(\omega_n)\}] = {} & \frac{(2\pi)^2}{4\beta E_{\rm c}}\sum_{\mathbf{i}} k_{\mathbf{i}}^2 + \pi g \sum_{|\mathbf{i}-\mathbf{j}|=a} |k_{\mathbf{ij}}| + \frac{\beta}{4E_{\rm c}}\sum_{\mathbf{i},n}\omega_n^2\phi_{\mathbf{i}}(\omega_n)\phi_{\mathbf{i}}(-\omega_n) \\ & + \frac{\beta g}{2}\sum_{|\mathbf{i}-\mathbf{j}|=a}(|\omega_{n+k_{\mathbf{ij}}}| + |\omega_{n-k_{\mathbf{ij}}}| - 2|\omega_{k_{\mathbf{ij}}}|)\times\ \phi_{\mathbf{ij}}(\omega_n)\phi_{\mathbf{ij}}(-\omega_n) + {\rm O}(\phi^4), \end{aligned} \tag{6}$$

where $\phi_{\mathbf{i}}(\tau) = \sum_n \phi_{\mathbf{i}}(\omega_n)\exp({\rm i}\omega_n\tau)$. Since the bare conductance is large, $g \gtrsim 1$, an expansion to quadratic order in the residual fluctuations is justified. The first two terms of equation (6) arise from finite-winding number (non-Gaussian) fluctuations, and directly lead to quantization of charge. The remaining terms in equation (6) arise from perturbation about the winding numbers. The competition of single-grain charging and hybridization at low enough temperatures can be seen in the partition function $Z_2 = \int {\rm D}\phi_1\,{\rm D}\phi_2\,\sum_{k_1,k_2}\exp(-S[k,\phi])$ of a simple two-grain system. Since the tunnelling term depends only on the phase difference between the two grains, we make a transformation to average phase $\phi_{\rm av} = (\phi_1+\phi_2)/2$, and relative phase $\phi = (\phi_1-\phi_2)$. Integrating out the relative phase gives a winding number dependent determinant. This we normalize against the determinant with no winding numbers:

$$\begin{aligned} \frac{{\rm Det}_\phi[k_{12}=0]}{{\rm Det}_\phi[k_{12}]} &= \frac{\prod_{n=1}^{\infty}\left[\frac{\pi n^2}{4\beta E_{\rm c}} + g|n|\right]}{\prod_{n=1}^{\infty}\left[\frac{\pi n^2}{4\beta E_{\rm c}} + \frac{g}{2}(|n+k_{12}| + |n-k_{12}| - 2|k_{12}|)\right]} \\ &= \prod_{n=1}^{|k_{12}|}\left[1+\frac{4g\beta E_{\rm c}}{n\pi}\right]\prod_{n=|k_{12}|+1}^{\infty}\left[\frac{\frac{\pi n^2}{4\beta E_{\rm c}} + gn}{\frac{\pi n^2}{4\beta E_{\rm c}} + g(n-|k_{12}|)}\right]. \end{aligned} \tag{7}$$

For $k_{12} \ll g\beta E_{\rm c}$ (a large g suppresses large values of k_{12}), the determinant in equation (7) can be simplified as

$$\begin{aligned} \frac{{\rm Det}_\phi[k_{12}=0]}{{\rm Det}_\phi[k_{12}]} &\sim \frac{1}{|k_{12}|!}\left(\frac{4g\beta E_{\rm c}}{\pi}\right)^{|k_{12}|}\prod_{n=|k_{12}|+1}^{4g\beta E_{\rm c}/\pi}\frac{n}{n-|k_{12}|} \\ &= \frac{(4g\beta E_{\rm c}/\pi)^{|k_{12}|}(4g\beta E_{\rm c}/\pi)!}{(4g\beta E_{\rm c}/\pi - |k_{12}|)!(|k_{12}|!)^2} \\ &\sim \exp\left[2|k_{12}|\ln\left(\frac{4ge\beta E_{\rm c}}{\pi|k_{12}|}\right)\right], \end{aligned} \tag{8}$$

where we used Stirling's formula for the factorials. This result implies that relative phase fluctuations enhance the tendency for phase slips between neighbouring grains. The effective

action for the two grain system takes the form

$$S[k,\phi_{\rm av}] = \frac{(2\pi)^2}{4\beta E_{\rm c}}\sum_i k_i^2 + \frac{2\beta}{4E_{\rm c}}\sum_n \omega_n^2\phi_{\rm av}(\omega_n)\phi_{\rm av}(-\omega_n) + \pi g|k_{12}|\left[1-\frac{2}{\pi g}\ln\left(\frac{4ge\beta E_{\rm c}}{\pi|k_{12}|}\right)\right]$$
$$= \frac{(2\pi)^2}{4\beta E_{\rm c}}\sum_i k_i^2 + \pi\gamma|k_{12}| + \frac{2\beta}{4E_{\rm c}}\sum_n \omega_n^2\phi_{\rm av}(\omega_n)\phi_{\rm av}(-\omega_n), \tag{9}$$

where the effective parameter

$$\gamma = g\left[1-\frac{2}{\pi g}\ln\left(\frac{4ge\beta E_{\rm c}}{\pi|k_{12}|}\right)\right] \tag{10}$$

is smaller than the renormalized tunnelling $g_{\rm ren} = g(1-1/(\pi dg)\ln(g\beta E_{\rm c}))$ that represents the conductance at large bare coupling g at not too low temperatures. The main interest of this paper is the regime $\gamma \lesssim 1$. If $1 \gtrsim \gamma \gg 1/(\beta E_{\rm c})$, most contribution to single charge excitations in the two-grain partition function comes from low winding number difference ($|k_{12}| \sim {\rm O}(1)$) between the grains. On the other hand, when $\gamma \lesssim 1/(\beta E_{\rm c})$, large winding number differences of the order of $\beta E_{\rm c}$ become important. Therefore in this regime, the charge is localized on either of the grains, with large charging energy of order $E_{\rm c}$.

It is important to obtain the charge representation by summing over the winding numbers with the use of the Poisson formula

$$Z_2 = \sum_{\{q_i\}}\int_{-\infty}^{\infty} {\rm D}\phi_{\rm av}\,\frac{{\rm d}x_1}{2\pi}\,\frac{{\rm d}x_2}{2\pi}{\rm e}^{{\rm i}2\pi\sum_i q_i x_i - S[x,\phi_{\rm av}]}. \tag{11}$$

Integrating out x_2 will yield the effective environment around x_1:

$$Z_2 \approx \sum_{q_1,q_2}\int {\rm d}x_1\int {\rm D}\phi_{\rm av}\,\exp\left\{-\frac{2\beta}{4E_{\rm c}}\sum_n \omega_n^2\phi_{\rm av}(\omega_n)\phi_{\rm av}(-\omega_n)\right\}$$
$$\times\left(\sqrt{\frac{\beta E_{\rm c}}{\pi}}\exp\left\{\frac{\gamma^2}{4\beta E_{\rm c}} - \beta E_{\rm c}q_2^2 - \frac{(2\pi)^2}{4\beta E_{\rm c}}x_1^2 + {\rm i}2\pi q_1 x_1\right\}\right.$$
$$\times[\Theta(x_1)\exp\{{\rm i}\gamma\beta E_{\rm c}q_2 - \pi\gamma x_1\} + \Theta(-x_1)\exp\{-{\rm i}\gamma\beta E_{\rm c}q_2 + \pi\gamma x_1\}]$$
$$\left.+\frac{\gamma/2\pi}{(\gamma/2)^2 + (q_2 + {\rm i}x_1\pi/\beta E_{\rm c})^2}\exp\left\{-\frac{2(2\pi)^2}{4\beta E_{\rm c}}x_1^2 + {\rm i}2\pi x_1(q_1+q_2)\right\}\right). \tag{12}$$

There are two qualitatively distinct contributions in equation (12). The first two terms represent isolated charging of grain 2. If γ were vanishingly small, this would be the only contribution. The last term represents hybridization of the two grains because of quantum tunnelling; the total charge $q_1 + q_2$ is shared between the grains and the charging energy $E_{\rm c}$ is reduced to $E_{\rm c}/2$. Finally, integration over x_1 gives the relative weights of the two processes in the partition function as $P_1 \approx \exp(-q_1^2\beta E_{\rm c})$ for isolated charging, and $P_2 \approx \frac{2\gamma}{\pi}\exp(-(q_1+q_2)^2\beta E_{\rm c}/2)$ for charging of the hybridized grains.

The treatment so far considers residual relative phase fluctuations only to Gaussian order. In the appendix, we present the results of path integral Monte Carlo calculations to support our basic idea of charge sharing over two grains, even at low temperatures, where non-Gaussian fluctuations are important. We also confirm that the temperature dependence of γ agrees well with equation (10) at not too low temperatures. As the temperature is decreased further, non-Gaussian fluctuations become important. The numerical calculations show that our physical picture, that a competition of charging and tunnelling effects determines whether the charge is shared between the two grains (with charging energy $E_{\rm c}/2$) or localized on a single grain,

still remains valid. More precisely, we find that the probability of sharing a charge between two grains (here γ) is an algebraically, and not exponentially, small function of temperature [2].

Consider now the case of N connected grains. Formally, it is simple enough to demonstrate annexation of a single grain into an N-site puddle. The proof is by induction. Suppose that an N-site puddle already exists (with statistical weight $\propto \gamma^N$). Integration over a string of $N-1$ contiguous neighbours of a grain **i** similarly gives a puddle of size N with charging energy E_c/N, and a weight $P_N \approx (\frac{2\gamma}{\pi})^{N-1} \exp(-(q_1 + \cdots + q_n)^2 \beta E_c/N)$. Such an expansion in γ only makes sense if $2\gamma/\pi < 1$. For $2\gamma/\pi > 1$, the optimum size of the puddle is divergent. Consider the action of a single grain coupled to this puddle:

$$\begin{aligned} S[k,\phi] = {} & \frac{(2\pi)^2 N k_N^2}{4\beta E_c} + \frac{(2\pi)^2 k_{N+1}^2}{4\beta E_c} + \pi g |k_{N,N+1}| \\ & + \frac{N\beta}{4E_c} \sum_n \omega_n^2 |\phi_N(\omega_n)|^2 + \frac{\beta}{4E_c} \sum_n \omega_n^2 |\phi_{N+1}(\omega_n)|^2 \\ & + \frac{\beta g}{2} \sum_n (|\omega_{n+k_{N,N+1}}| + |\omega_{n-k_{N,N+1}}| - 2|k_{N,N+1}|) |\phi_{N,N+1}(\omega_n)|^2. \end{aligned} \tag{13}$$

In terms of the centre of mass coordinate

$$\phi_{\mathrm{av}} = \frac{N\phi_N + \phi_{N+1}}{N+1}, \tag{14}$$

and relative coordinate

$$\phi = \phi_N - \phi_{N+1}, \tag{15}$$

the action takes the form

$$\begin{aligned} S[k,\phi] = {} & \frac{(2\pi)^2 N k_N^2}{4\beta E_c} + \frac{(2\pi)^2 k_{N+1}^2}{4\beta E_c} + \pi g |k_{N,N+1}| \\ & + \frac{\beta}{4E_c} \sum_n \omega_n^2 \left[(N+1) |\phi_{\mathrm{av}}(\omega_n)|^2 + \frac{N}{N+1} |\phi(\omega_n)|^2 \right] \\ & + \frac{\beta g}{2} \sum_n (|\omega_{n+k_{N,N+1}}| + |\omega_{n-k_{N,N+1}}| - 2|k_{N,N+1}|) |\phi(\omega_n)|^2. \end{aligned} \tag{16}$$

Integrating out the relative phase renormalizes the bare coupling g in a manner similar to that in equation (10),

$$\gamma_{N,N+1} = g \left[1 - \frac{2}{\pi g} \ln \left(\frac{2eg\beta E_c}{\pi |k_{N,N+1}|} \frac{N}{N+1} \right) \right]. \tag{17}$$

Note that the relevant $\gamma_{N,N+1}$ determining annexation of a single grain into an N-site puddle is not too different from γ for a two grain system obtained in equation (10). Accordingly, the condition for the suppression of large winding number difference changes from $\gamma \gg 1/(\beta E_c)$ for two grains to the condition $\gamma \gg N/(\beta E_c)$ for N grains. Performing the summation over k_N and k_{N+1} in equation (16) using the Poisson summation formula again yields two terms that correspond to separate charging of the puddle and grain, and charging of the larger $(N+1)$-site puddle. The criterion for annexation is

$$\frac{2\gamma_{N,N+1}}{\pi} \exp[-\beta E_c/(N+1)] > \exp[-\beta E_c/N]. \tag{18}$$

[2] In equation (12) if we choose to normalize the partition function by the partition function for $(q_1 - q_2) = 0$, the factor γ in the numerator of the second term of the equation should be replaced by γ^2. This does not affect our physical picture of charge sharing.

So far we have obtained the effective environment of a site **i** by integrating out a sequence of $N-1$ contiguous neighbours. Integrating over such 'strings' is somewhat different from the actual requirement that one should consider an arbitrary puddle with N sites, and integrate over all N phases. Since the number of bonds exceeds the number of sites in two and three dimensions, it would be incorrect to consider the phase differences between bonds as independent variables. The maximum number of independent phase differences in a puddle of N sites is $N-1$. Starting from an arbitrary site in the puddle, a non-self-intersecting string of $N-1$ bonds spans all N sites. The string, however, is not unique, hence in the partition function Z_N for N coupled sites, one must consider all possible self-avoiding string configurations of $N-1$ links. From the theory of self-avoiding random walks [15], it is known that the degeneracy $\mathcal{N}$ of such configurations is

$$\mathcal{N}(N) \sim \begin{cases} (N-1)^{1/6}\tilde{z}_3^{N-1}, & d=3 \\ (N-1)^{1/3}\tilde{z}_2^{N-1}, & d=2, \end{cases} \tag{19}$$

where $\tilde{z}_d$ is an effective coordination number that depends on the dimensionality and the arrangement of grains. For a simple cubic lattice in three dimensions, $\tilde{z}_3 = 4.68$, slightly less than 6, which is the actual coordination number. Thus the contribution of an N-site puddle to the partition function, say in three dimensions, is

$$Z_N \approx (N-1)^{1/6}\sqrt{\frac{\beta E_c}{\pi N}}\left(\frac{2\gamma\tilde{z}_3}{\pi}\right)^{N-1}\exp\left(-\frac{\beta E_c}{N}q_N^2\right) \times \int \mathrm{D}\phi \exp\left(-\frac{N\beta}{4E_c}\sum_n \omega_n^2\phi(\omega_n)\phi(-\omega_n)\right). \tag{20}$$

The optimum size of the puddle is reached when $N = N_* \approx \sqrt{\frac{\beta E_c q_{N_*}^2}{\ln(\pi/2\tilde{z}_3\gamma)}}$, and the dominant contribution to the partition function is

$$Z_{N_*} \approx \frac{\pi}{2\gamma\tilde{z}_3}\exp\left(-2\sqrt{\beta E_c q_{N_*}^2 \ln(\pi/2\tilde{z}_3\gamma)}\right). \tag{21}$$

This result is valid under the condition $1 \gtrsim \gamma \gg 1/\sqrt{\beta E_c}$.

We now have the necessary ingredients for calculating the conductivity σ and tunnelling density of states $\nu_{\mathbf{i}}$ from equations (3) and (4). Calculation of the conductivity σ using equation (3) requires evaluation of a two-point phase correlation function $\tilde{\Pi}_{\mathbf{i},\mathbf{i}+\mathbf{a}}$,

$$\tilde{\Pi}_{\mathbf{i},\mathbf{i}+\mathbf{a}} = \langle\exp(-\mathrm{i}(\tilde{\phi}_{\mathbf{i},\mathbf{i}+\mathbf{a}}(\tau) - \tilde{\phi}_{\mathbf{i},\mathbf{i}+\mathbf{a}}(0)))\rangle. \tag{22}$$

The two points **i** and **i** + **a** should be chosen to lie in different puddles, for if they lie within the same puddle, $\tilde{\Pi}$ would simply describe the fluctuation of charge distribution inside a puddle; this contributes little to the conductivity σ. This simplifies the evaluation of the two-point phase correlation function to a product of two one-point phase correlation functions, $\tilde{\Pi}_{\mathbf{i},\mathbf{i}+\mathbf{a}} \approx \langle\exp(-\mathrm{i}(\tilde{\phi}_{\mathbf{i}}(\tau) - \tilde{\phi}_{\mathbf{i}}(0)))\rangle\langle\exp(\mathrm{i}(\tilde{\phi}_{\mathbf{i}+\mathbf{a}}(\tau) - \tilde{\phi}_{\mathbf{i}+\mathbf{a}}(0)))\rangle$. The averaging in equation (22) should be performed over winding numbers $\{k_{\mathbf{i}}\}$ as well as the phase fluctuations $\{\phi_{\mathbf{i}}\}$. The AES action in equation (13) after integrating over the relative residual phase fluctuations then takes the form

$$S[\{k_{\mathbf{i}}\};\{\phi_{\mathbf{i}}(\omega_n)\}] = \frac{(2\pi)^2}{4\beta E_c}\sum_{\mathbf{i}} N_{\mathbf{i}}k_{\mathbf{i}}^2 + \pi\gamma\sum_{|\mathbf{i}-\mathbf{j}|=a}|k_{\mathbf{i}} - k_{\mathbf{j}}| + \frac{\beta}{4E_c}\sum_n N_{\mathbf{i}}\omega_n^2\phi_{\mathbf{i}}(\omega_n)\phi_{\mathbf{i}}(-\omega_n) + \cdots. \tag{23}$$

Upon performing the average, we obtain an expansion in increasing puddle size:

$$\tilde{\Pi}_{\mathbf{i},\mathbf{i+a}} \approx \sum_{\{N_\mathbf{i}\}} \left(\frac{2\gamma\tilde{z}_d}{\pi}\right)^{N_\mathbf{i}+N_\mathbf{i+a}-2} \exp\left\{-E_\mathrm{c}\tau\left(\frac{1}{N_\mathbf{i}}+\frac{1}{N_\mathbf{i+a}}\right)\right\} \times \sum_{\{q_N\}} \exp\left\{2\tau E_\mathrm{c}\left(\frac{q_{N_\mathbf{i}}}{N_\mathbf{i}}-\frac{q_{N_\mathbf{i+a}}}{N_\mathbf{i+a}}\right)-\beta E_\mathrm{c}\left(\frac{q_{N_\mathbf{i}}^2}{N_\mathbf{i}}+\frac{q_{N_\mathbf{i+a}}^2}{N_\mathbf{i+a}}\right)\right\}. \tag{24}$$

To calculate the conductivity σ given by equation (3), we make the analytic continuation $\Omega_n \to -\mathrm{i}\omega + \epsilon$, and deform [3] the contour of integration in the following manner: $(0,\beta) \to (0,\mathrm{i}\infty) + (\mathrm{i}\infty, \mathrm{i}\infty+\beta) + (\mathrm{i}\infty+\beta, \beta)$. For dc conductivity, we expand equation (3) for small ω, and take the limit $\omega \to 0$. Performing the integration yields the conductivity

$$\sigma \sim 2ga^{2-d} \sum_{N_\mathbf{i},N_\mathbf{i+a}} \sum_{q_{N_\mathbf{i}},q_{N_\mathbf{i+a}}} \left(\frac{2\gamma\tilde{z}_d}{\pi}\right)^{N_\mathbf{i}+N_\mathbf{i+a}-2} \exp\left(-\beta E_\mathrm{c}\left(\frac{(q_{N_\mathbf{i}}-1)^2}{N_\mathbf{i}}+\frac{(q_{N_\mathbf{i+a}}+1)^2}{N_\mathbf{i+a}}\right)\right). \tag{25}$$

Most of the contribution to equation (25) comes from two single-charge configurations $(q_{N_\mathbf{i}}, q_{N_\mathbf{i+a}}) = (1,0)$, or $(0,-1)$. In the former configuration, conductivity is dominated by $(N_\mathbf{i}, N_\mathbf{i+a}) = (1, N_*)$, while in the latter configuration, conductivity is dominated by $(N_\mathbf{i}, N_\mathbf{i+a}) = (N_*, 1)$, and $N = N_* \approx \sqrt{\frac{\beta E_\mathrm{c}}{\ln(\pi/2\tilde{z}_d\gamma)}}$ as usual. The result is

$$\sigma \sim \frac{1}{\tilde{z}_d} a^{2-d} \exp\left(-2\sqrt{\beta E_\mathrm{c} \ln(\pi/2\gamma\tilde{z}_d)}\right). \tag{26}$$

We can similarly obtain the tunnelling density of states:

$$\nu(\varepsilon) \approx \frac{\pi\nu_0}{2\gamma\tilde{z}_d} \cosh(\beta\varepsilon) \exp\left(-2\sqrt{\beta E_\mathrm{c} \ln(\pi/2\gamma\tilde{z}_d)}\right). \tag{27}$$

3. Conclusion

We propose that our simple model of a regular array may explain soft activation behaviour observed in real granular metals [4, 6–8]. In real granular metals, inter-grain tunnelling may vary strongly between grains, but even in the presence of disorder, our physical mechanism could be applicable. Firstly, for weak disorder, suppose the inter-grain coupling for the ith tunnelling link has a distribution $\gamma_i = \gamma^{1+\epsilon_i}$. Then for an N-site puddle, since the ϵ_i are random, the tunnelling term $\prod_i \gamma_i \sim \gamma^{N+\sum_i \epsilon_i} \sim \gamma^N$ is not seriously modified, and our conclusions hold. Secondly, as discussed in the context of granular superconductors [16], theoretical calculations based on regular Josephson arrays seem to be relevant. The reason is that even for a wide distribution of couplings, only a narrow range of couplings is relevant, since (a) the extremely weak links can effectively be disregarded and (b) for links that are much stronger than average, one can approximate the connected grains as one single grain. While the tunnelling probability changes exponentially with length, the charging energy changes only linearly, so the variation of charging energies is relatively small. The system then effectively consists of such renormalized 'grains' linked by tunnelling of similar magnitude. Thirdly, if conduction occurs through a few 1D paths, our result, being dimensionality independent, still applies. The observations [7, 8], according to our picture, are robust even upon application of strong magnetic fields ($\gtrsim$10 T) and are independent of dimensionality [8]. Nevertheless, further work needs to be done to understand properly granular metals with strong variation in inter-grain couplings.

The AES approach we use, views conduction as a Fermi golden-rule type incoherent tunnelling process. The obvious difference between our picture of soft activation and the

Efros–Shklovskii [17] (ES) theory, which also gives a similar temperature dependence of conductivity, is the on-site charging energy cost E_c in our model and lack of thereof in ES theory. Furthermore, the mutual interaction of charges (and excitonic effects) on widely separated grains plays no significant role in our analysis unlike in ES theory. Since the soft activation mechanism involves only nearest-neighbour hopping in comparison with long-distance variable range hopping, the magnetoresistance of soft activation here is expected to be very weak, which is consistent with experiments [8]. Another possibility [4] considered in the literature suggests that the observed soft activation could be an artifact of a special distribution of grain sizes. Such a hope is belied by observation [7] of the same soft activation in samples with a very narrow distribution of grain sizes. Also if we accept the conduction process as proceeding through tunnelling of charge between neighbouring grains, there would be little likelihood of finding the percolation paths in the wide range of temperatures through appropriately sized grains, should they exist, as neighbours.

The relevance of our results as well as references [3, 12] should be explored beyond carefully prepared granular arrays. Recently, a logarithmic temperature dependence of conductivity in strong magnetic fields [18, 19] and granular (or domain) structure [20] has been observed in certain underdoped cuprates. The insulating phase (even more underdoped) in the same materials exhibited the soft activation behaviour [21], which may be due to the mechanism proposed in this paper.

In conclusion, our analysis of transport in granular arrays at not too low temperatures $T \gg \max(\delta, g\delta)$ in the framework of the AES approach shows that the transitions from a logarithmic temperature dependence of conductivity for strong inter-grain coupling ($g \gg 1$) to the soft activation behaviour for intermediate coupling ($g \gtrsim 1, 1 \gtrsim \gamma \gg 1/\sqrt{\beta E_c}$) and further to the hard activation behaviour for weak coupling $g \ll 1$ can be understood as arising from the competition between Coulomb blockade and tunnelling. This analysis is strictly valid for regular arrays and may be considered for experimental systems [4, 6–8].

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Appendix

Note added in proof. The main idea of our paper is that charge sharing among several grains occurs at certain low temperatures, and the probability of sharing a charge between two grains is not an exponentially small function of temperature. In this appendix we discuss various arguments (in addition to the calculation in the main text) in support of this main idea, which we developed after the original manuscript was submitted. We hope to publish a more detailed discussion elsewhere.

We discuss in detail the situation of two connected grains to demonstrate again the essential physics. For two grains, the AES action can always be expressed in terms of the average phase $\phi_{av} = (\phi_1 + \phi_2)/2$ and relative phase $\phi = (\phi_1 - \phi_2)$, where ϕ_1 and ϕ_2 are the phases of the first and second grains.

$$S = \frac{1}{4E_c}\int_0^\beta d\tau \left(\frac{d\phi_1}{d\tau}\right)^2 + \frac{1}{4E_c}\int_0^\beta d\tau \left(\frac{d\phi_2}{d\tau}\right)^2 + S_t(\phi_1 - \phi_2)$$
$$= \frac{1}{4(E_c/2)}\int_0^\beta d\tau \left(\frac{d\phi_{av}}{d\tau}\right)^2 + \frac{1}{4(2E_c)}\int_0^\beta d\tau \left(\frac{d\phi}{d\tau}\right)^2 + S_t(\phi). \qquad (28)$$

The part of the action for the average phase is trivial and is easily transformed to the charge representation. In doing so it is necessary to satisfy carefully correct Matsubara boundary conditions of the original fields ϕ_1 and ϕ_2.

The crucial question is what the minimum charging energy of two grains is. Is the minimum charging energy $E_c/2$ or still E_c as for a single grain? For a two-grain system, this issue can be addressed by considering the phase correlation function for one of the grains,

$$\begin{aligned} C_1(\tau) &= \langle \cos(\phi_1(\tau) - \phi_1(0)) \rangle \\ &= \langle \cos(\phi_{\rm av}(\tau) - \phi_{\rm av}(0)) \rangle \langle \cos((\phi(\tau) - \phi(0))/2) \rangle. \end{aligned} \tag{29}$$

The part of the action for the average phase $\phi_{\rm av}$ corresponds to the charging energy $E_c/2$ for the total charge $(q_1 + q_2)$, quantized and equal to one. From this part, $C_1(\tau)$ gets a contribution $\exp(-E_c\tau/2)$. If g were zero, then the relative phase contribution to $C_1(\tau)$ is also $\exp(-E_c\tau/2)$, so that $C_1(\tau) = \exp(-E_c\tau)$, in accordance with our expectation for isolated grains. For a finite value of g, we demonstrate below that the correlation functions of the relative phase fluctuations at large τ decrease only algebraically as a function of temperature, and do not show a hard Coulomb gap [23][3]. The gaplessness of the relative phase fluctuations unambiguously proves that the minimum charging energy of the two grains is halved, $E_c/2$, and associated only with the average phase. If instead the relative phase correlator were gapped, with some effective charging energy $E_c^\star$, then the correlation function would decrease exponentially at long-τ with the corresponding charging energy. Therefore the question about the effective charging energy is equivalent to the question of considering the long-τ asymptotics (or equivalently low temperatures) of the correlation function $C_1(\tau)$. The last statement is of a general character, since the long-time (or low temperature) asymptotics always reveals the lowest energy excitations (or configurations of the ϕ-field) of the system. Namely, if the charge gap exists, this will become evident as an exponential decay of the corresponding correlation function at long times. In the literature, the charge gap is occasionally related to the amplitude of Coulomb blockade oscillations as a function of a gate voltage on a grain. In our case, the charge gap is the cost of putting one excess charge on a grain. These two definitions are not necessarily the same. For calculating the conductivity of the granular system, our definition of the charge gap is the appropriate one.

Let us consider the relative phase correlator

$$C(\tau) = \langle \cos(\phi(\tau) - \phi(0)) \rangle, \tag{30}$$

which has been extensively studied in the literature, so a comparison with known results is possible. In addition, $C(\tau) \leqslant \langle \cos(\phi(\tau) - \phi(0))/2 \rangle$, so if we can show that the large τ behaviour of $C(\tau)$ does not have a hard gap, then it is also true for $\langle \cos(\phi(\tau) - \phi(0))/2 \rangle$. We claim (and we are not the first ones) that the correlation function $C(\tau)$ of the single-phase action decays, in fact, as a power-law, $(T/T_*)^2/\sin^2(\pi T\tau)$, at very large τ and *not exponentially*. Here T_* is an energy scale exponentially small in g. Thus the correlation function of relative phase fluctuations is not gapped for long times. This is a crucial point because a temperature dependence of $C(\tau)$ that is not exponentially small in temperature at large τ invariably leads to a soft activation behaviour of conductivity at low temperatures (see concluding remarks in this section). Several arguments based on general results of statistical physics and mesoscopics as well as our numerical Monte Carlo simulations prove beyond doubt that the correlation function $C(\tau)$ decays algebraically (proportional to $1/\sin^2(\pi T\tau)$). First, note that the action for relative phase fluctuations is a one-dimensional field theory with

[3] This work proposed an insulating phase with hard activation gap at low temperatures. It seems that the approximation of non-interacting instantons used in the work [23] (and in some others, e.g. [1]) may not be valid, as for instance has been discussed in [14].

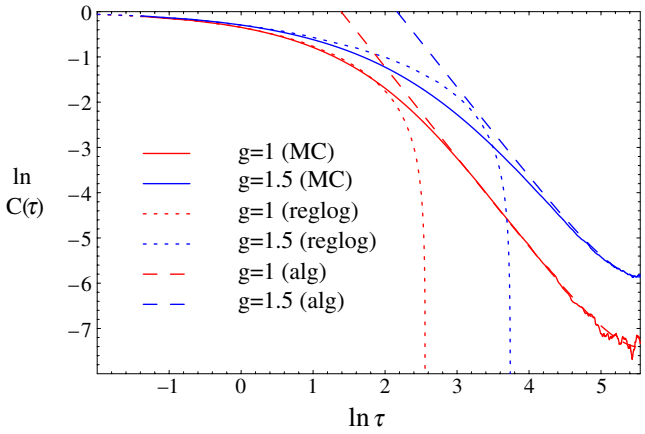

Figure A.1. $\ln C(\tau)$ versus $\ln \tau$ for $g = 1$ (red) and $g = 1.5$ (blue), at $\beta = 768$. The data (solid curves) show a crossover from logarithmic behaviour C^{reglog} (dotted curves) to the power-law behaviour C^{alg} (dashed). Discrepancies at small τ are due to discretization error. Note that C^{reglog} is not exactly a power law (or a straight line on a log–log plot): the cosec squared flattens out at $\tau \sim \beta/2$.

a long-range interaction, $gT^2/\sin^2(\pi T\tau)$, in imaginary time. A general theorem of statistical physics due to Griffiths [22] states that the correlation function $C(\tau)$ cannot decay faster than the interaction, $gT^2/\sin^2(\pi T\tau)$. The exponential decay is much faster than algebraic decay and therefore not possible. Second, it is widely recognized in the mesoscopics literature that at low temperatures the tunnelling to a quantum dot is dominated by so-called inelastic (or elastic) cotunnelling processes [24, 25]. In the case of two grains, cotunnelling processes which are second order processes in the conductance g correspond to the creation of electron–hole pairs on both grains. These processes are the lowest-energy gapless processes which can be closely associated with long-τ behaviour of $C(\tau)$, once again this demonstrates that the charge gap at low temperatures is effectively zero. In fact, the picture of the charge sharing can be equally well discussed in terms of the balance between Coulomb blockade and cotunnelling processes for a finite set of grains. Our results concerning cotunnelling processes are somewhat non-trivial, because we describe these processes in terms of the parameter $\gamma(T)$ for $g \gg 1$ (unlike the originally considered case of $g \ll 1$ of [24, 25]). Third, we undertook numerical simulations of the single-phase AES action using the path integral Monte Carlo method. It is possible to calculate directly by this method, without any approximations, not only the correlation function $C(\tau)$ but also the parameter γ as a function of g and T. Numerical results show clearly that the correlation function behaves as $(T/T_*)^2/\sin^2(\pi T\tau)$ in the large-τ limit. Note that a rough estimate of $C(\tau)$ as $C(\tau) \sim [\sum_k \cos(2\pi kT\tau)\exp(-S(k))]/\exp(-S(k))$ using equation (9) gives $C(\tau) \sim \sinh^2(\pi\gamma/2)/[\sinh^2(\pi\gamma/2)+\sin^2(\pi T\tau)]$. For small values of γ, the Gaussian approximation is inaccurate, nevertheless the Lorentzian long time behaviour of $C(\tau)$ inferred from equation (9) clearly anticipates the $(T/T_*)^2/\sin^2(\pi T\tau)$ result of exact numerical calculations, with $\gamma \sim T/T_* \ll 1$ and large τ.

In what follows we summarize the results of the path integral Monte Carlo simulations (the description of the method and further results will be published elsewhere [26]). In figure A.1 we present the correlation function $C(\tau)$ for $g = 1$ and 1.5. One can see clearly that at short imaginary times—τ, $C(\tau) = 1 - 1/(\pi g)\ln(gE_c/2T)$, which is consistent with g_{ren} calculated in equation (5) ignoring winding numbers and various others [3]. At long times—τ, in figure A.1, the correlation function decays as $1/\sin^2(\pi T\tau)$.

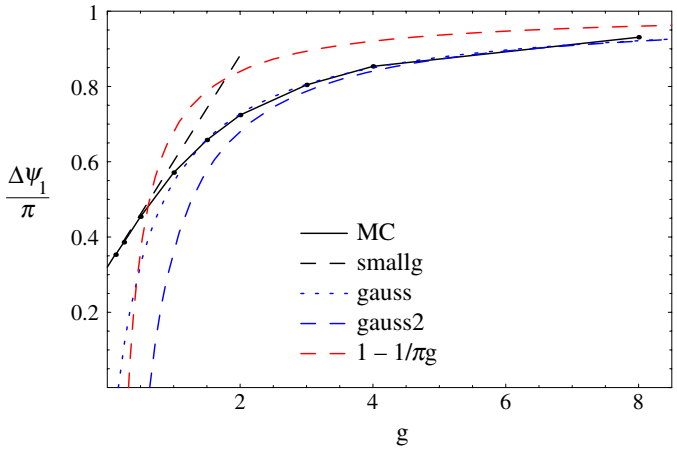

Figure A.2. $\Delta\psi_1(g)$, defined as the derivative with respect to g of the normalized action (see equation (32)), plotted against g for $\beta = 16$. Solid black curve: Monte Carlo. Dashed black line: small-g perturbation theory. Dotted blue curve: $\Delta\psi_1^{\text{gauss}}$, using the Gaussian approximation of equation (31). Dashed blue curve: $\Delta\psi_1^{\text{gauss2}}$, using equation (10) which can be deduced from equation (31) using Stirling's approximation. Dashed red curve: $(1 - 1/\pi g)$, i.e. the derivative of g_{ren} with respect to g.

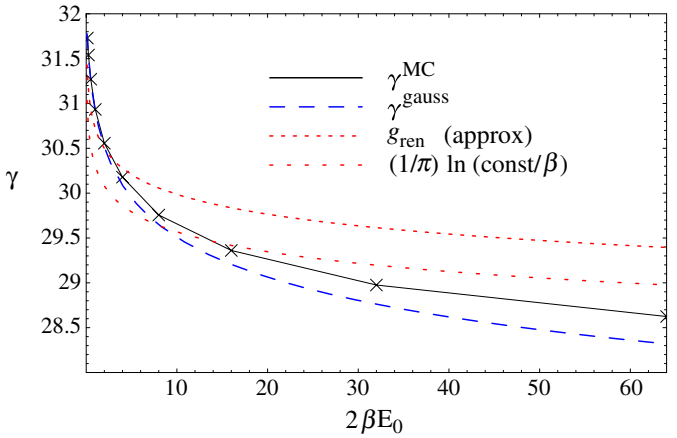

Figure A.3. γ versus β for $g = 32$. Solid black curve: Monte Carlo calculation. Dashed blue curve: $\gamma(T)$ in equation (31) obtained by considering Gaussian fluctuations. Top red curve: $g_{\text{ren}}(T)$ extracted from $C(\tau)$ with τ proportional to β. Bottom red curve: $g_{\text{ren}}(T)$ with adjusted cutoff, $g_{\text{ren}}(T) = [g - (1/\pi)\ln(\text{constant} \times E_c/T)]$. Clearly, $(1/\pi)\ln T$ is not in good agreement with Monte Carlo results for any adjustment of the cutoff. The Gaussian approximation gives a much better agreement with Monte Carlo results.

The detailed behaviour of the parameter γ is given in figure A.2 as a function of g and in figure A.3 as a function of temperature T. The exact calculation of the determinant of the residual fluctuations in the Gaussian approximation gives instead of $\pi\gamma|k_{12}|$ in equation (9) the following expression:

$$\pi\gamma|k| = \pi g|k| - \ln \frac{\Gamma\left(1 + k + \frac{x - \sqrt{x}\sqrt{x+4k}}{2}\right)\Gamma\left(1 + k + \frac{x + \sqrt{x}\sqrt{x+4k}}{2}\right)}{\Gamma(1+k)^2\Gamma(1+x)}, \tag{31}$$

where $x = \frac{gE_c}{\pi T}$ and for brevity we denote the relative winding number $k_{12} \equiv k$ as k. This expression is more precise than the expression (see equation (10)) calculated using Stirling's approximation.

Figure A.2 compares the derivative of the action,

$$\Delta\Psi_1(g) = \partial[S(k=1,g) - S(k=0,g)]/\partial g, \tag{32}$$

evaluated numerically, with various analytic approximations. The blue dotted curve is the Gaussian approximation of equation (31) and the blue dashed curve is equation (10), which can be shown to follow from equation (31) using Stirling's approximation for the gamma functions. The red dashed curve corresponds to setting $\gamma(T) = g_{\rm ren}$.

Figure A.3 compares $\gamma(T)$, calculated numerically, with the Gaussian approximation in equation (31) (blue dashed curve), and $g_{\rm ren}(T)$ calculated from $C(\tau)$, $\tau \propto \beta$, (top red curve) and $g_{\rm ren}(T)$ with an adjustable cutoff (lower red curve), as a function of temperature.

We thus observe that the renormalizations of the quantities $g_{\rm ren}$ and γ as a function of temperature are *different* for large $g \gg 1$. Although this observation is not essential on its own for the charge sharing mechanism, the numerical simulation shows directly that the renormalization of γ is stronger (see below). The stronger renormalization of γ in comparison with $g_{\rm ren}$ should presumably be associated with nearly zero-modes of residual fluctuations which exist around winding number trajectories. Namely, the Gaussian fluctuations are stronger around winding number trajectories than around non-winding ($k = 0$) trajectories because the square averaged fluctuations for zero modes (for $n \leqslant k$) are much stronger $\langle\phi_n^2\rangle_{\rm zm} = 4E_c/(\pi T)$ than $\langle\phi_n^2\rangle \sim 1/g$ for simple Gaussian residual fluctuations. Note that zero-mode fluctuations need to be considered beyond Gaussian approximation for $E_c \gg T$, because $\langle\phi_n^2\rangle_{\rm zm}$ becomes easily much larger than $(2\pi)^2$. Therefore fluctuations beyond logarithmic renormalizations are naturally expected and do occur as seen numerically.

We end with two remarks. Since the temperature dependence of the relative phase correlator at large τ is not an exponentially small function of temperature, but only a power law, optimizing the probability of a charge shared among N grains, $P_N \sim \gamma(T)^{N-1}\exp(-E_c/NT) \approx \exp(-2\sqrt{E_c\ln(\gamma^{-1})/T})$, gives a temperature dependence of the exponent that is always weaker than Arrhenius' law, $P \sim \exp(-E_c^*/T)$. Thus the temperature dependence of the optimum probability P_N is the soft activation behaviour. Second, it has not escaped our attention that even for $g < 1$, inelastic cotunnelling processes should make the charge sharing possible [26].

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